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Necessary and sufficient conditions for inclusion relations for absolute summability

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Abstract. We obtain a set of necessary and sufficient conditions for $|\overline{N}, p_n|_k$ to imply $|\overline{N}, q_n|_s$ for $1 < k \leq s < \infty$. Using this result we establish several inclusion theorems as well as conditions for the equivalence of $|\overline{N}, p_n|_k$ and $|\overline{N}, q_n|_s$.

Keywords. Absolute summability; weighted mean matrix; Cesáro matrix.

In 1994, Sarıgöl [6] obtained necessary and sufficient conditions for $|\overline{N}, p_n|_k$ to imply $|\overline{N}, q_n|_s$ for $1 < k \leq s < \infty$, using the definition that a series $\sum a_k$ is summable in $|\overline{N}, p_n|_k$ iff

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n} \right)^{k-1} |\Delta T_n|^k < \infty, \tag{1}$$

where T_n is the nth term of the $|\overline{N}, p_n|$ transform of the sequence of partial sums of the series $\sum a_n$.

As pointed out by the first author in [3] the correct condition is

$$\sum_{n=1}^{\infty} n^{k-1} |\Delta T_n|^k < \infty. \tag{2}$$

In this paper we obtain appropriate necessary and sufficient conditions for $|\overline{N}, p_n|_k$ summability to imply that of $|\overline{N}, q_n|_s$ for $1 < k \leq s < \infty$. As in [6] we make use of a result of Bennett [1], who has obtained necessary and sufficient conditions for a factorable matrix to map $\ell^k \to \ell^s$. A factorable matrix A is one in which each entry $a_{nk} = b_n c_k$. Weighted mean matrices are factorable.

It will not be possible to extended our result by replacing $(\overline{N}, q_n)$ by a triangular matrix A, since necessary and sufficient conditions are not known for an arbitrary triangular matrix B to map: $\ell^k \to \ell^s$.

However, if $k = 1$, then the necessary and sufficient conditions can be obtained. Such a result is the special case, by setting each $\lambda_n = 1$ of Theorem 2.1 of Rhoades and Savaş [4].

Our main result is the following:

Theorem. *Let $\{p_n\}$ and $\{q_n\}$ be positive sequences, $1 < k \le s < \infty$. Then $|\overline{N}, p_n|_k \Rightarrow |\overline{N}, q_n|_s$ iff*

(i)
$$n^{(1/k-1/s)}\frac{q_n P_n}{p_n Q_n} = O(1) \tag{3}$$

and

(ii)
$$\left(\sum_{n=m}^{\infty}\left(n^{1-1/s}\frac{q_n}{Q_n Q_{n-1}}\right)^s\right)^{1/s}\left(\sum_{v=1}^{m}\left|Q_v - \frac{q_v P_v}{p_v}\right|^{k^*}\left(\frac{1}{v}\right)\right)^{1/k^*} = O(1), \tag{4}$$

where k^* denotes the conjugate index of k i.e., $1/k + 1/k^* = 1$.

Proof. Let (x_n) and (y_n) denote the nth terms of the $|\overline{N}, p_n|$ and $|\overline{N}, q_n|$, transforms respectively of $s_n = \sum_{i=0}^{n} a_i$. Then

$$X_n := x_n - x_{n-1} = \frac{p_n}{P_n P_{n-1}}\sum_{v=1}^{n-1} P_{v-1} a_v \tag{5}$$

and

$$Y_n := y_n - y_{n-1} = \frac{q_n}{Q_n Q_{n-1}}\sum_{v=1}^{n} Q_{v-1} a_v. \tag{6}$$

Solving (5) for a_n gives

$$\frac{P_n P_{n-1} X_n}{p_n} = \sum_{v=1}^{n} P_{v-1} a_v,$$

$$\frac{P_{n-1} P_{n-2} X_{n-1}}{p_{n-1}} = \sum_{v=1}^{n} P_{v-1} a_v.$$

Thus

$$\frac{P_n P_{n-1} X_n}{p_n} - \frac{P_{n-1} P_{n-2} X_{n-1}}{p_{n-1}} = P_{n-1} a_n,$$

or

$$a_n = \frac{P_n X_n}{p_n} - \frac{P_{n-2} X_{n-1}}{p_{n-1}}. \tag{7}$$

Substituting (7) into (6) we have

$$Y_n = \frac{q_n}{Q_n Q_{n-1}}\sum_{v=1}^{n} Q_{v-1}\left[\frac{P_v X_v}{p_v} - \frac{P_{v-2} X_{v-1}}{p_{v-1}}\right]$$

$$= \frac{q_n}{Q_n Q_{n-1}}\left[\sum_{v=1}^{n}\frac{Q_{v-1} P_v X_v}{p_v} - \sum_{v=1}^{n}\frac{Q_{v-1} P_{v-2} X_{v-1}}{p_{v-1}}\right]$$

$$= \frac{q_n}{Q_n Q_{n-1}} \left[\sum_{v=1}^{n} \frac{Q_{v-1} P_v X_v}{p_v} - \sum_{i=0}^{n-1} \frac{Q_i P_{i-1} X_i}{p_i} \right]$$

$$= \frac{q_n}{Q_n Q_{n-1}} \left[\frac{Q_{n-1} P_n X_n}{p_n} + \sum_{v=1}^{n-1} (P_v Q_{v-1} - Q_v P_{v-1}) \frac{X_v}{p_v} \right].$$

But

$$P_v Q_{v-1} - Q_v P_{v-1} = P_v(Q_v - q_v) - Q_v(P_v - p_v)$$
$$= -q_v P_v + p_v Q_v.$$

Therefore

$$Y_n = \frac{q_n P_n X_n}{p_n Q_n} + \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n-1} \left(Q_v - \frac{q_v P_v}{p_v} \right) X_v.$$

Define

$$Y_n^* = n^{1-1/s} Y_n, \quad X_n^* = n^{1-1/k} X_n.$$

Then

$$Y_n^* = n^{1-1/s} \left[\frac{q_n P_n}{p_n Q_n} \left(\frac{X_n^*}{n^{1-1/k}} \right) \right.$$

$$\left. + \frac{q_n}{Q_n Q_{n-1}} \sum_{v=1}^{n-1} \left(Q_v - \frac{q_v P_v}{p_v} \right) \left(\frac{X_v^*}{v^{1-1/k}} \right) \right].$$

Thus, $Y_n^* = \sum_{v=1}^{n} a_{nv} X_v^*$, where

$$a_{nv} = \begin{cases} \dfrac{n^{1-1/s} q_n}{Q_n Q_{n-1}} \dfrac{\left(Q_v - \frac{q_v P_v}{p_v} \right)}{v^{1-1/k}}, & 1 \le v < n \\[2ex] \dfrac{n^{1/k-1/s} q_n P_n}{p_n Q_n}, & v = n \\[2ex] 0, & v > n \end{cases} \tag{8}$$

Then $|\overline{N}, p_n|_k \Rightarrow |\overline{N}, q_n|_s$ is equivalent to

$$\sum |X_n^*|^k < \infty \quad \Rightarrow \quad \sum |Y_n^*|^s < \infty; \quad \text{i.e.,} \quad A : \ell^k \to \ell^s,$$

where A is the matrix whose entries are defined by (8). We may write $A = B + C$, where $b_{nv} = a_{nv}$ for $1 \le v < n; b_{nv} = 0$, otherwise, and C is the diagonal matrix with $c_{nn} = a_{nn}$. Omitting the first row of B, which contains all zeros, what remains is a factorable matrix.

From Theorem 2(ii) of [1] a factorable matrix with nonzero entries $b_n c_v$, is a bounded operator from ℓ^k to ℓ^s iff

$$\left(\sum_{n=m}^{\infty} b_n^s \right)^{1/s} \left(\sum_{v=1}^{m} c_v^{k^*} \right)^{1/k^*} = O(1), \tag{9}$$

where k^* is conjugate index to k.

Applying (9) to B, and using (8), we have that $B : \ell^k \to \ell^s$ iff

$$\left(\sum_{n=m+1}^{\infty} \left(\frac{n^{1-1/s}q_n}{Q_n Q_{n-1}} \right)^s \right)^{1/s} \left(\sum_{v=1}^{m} \left| Q_v - \frac{q_v P_v}{p_v} \right|^{k^*} \left(\frac{1}{v} \right) \right)^{1/k^*} = O(1). \qquad (10)$$

Since $k \le s, C : \ell^k \to \ell^s$, i.e.,

$$\left(\sum_{n=1}^{\infty} |c_{nn}s_n|^s \right)^{1/s} < \infty \qquad (11)$$

for every $\{s_n\} \in \ell^k$. But (11) implies that $\{c_{nn}\} \in \ell^{s^*}$, where s^* is the conjugate of s. In particular $\{c_{nn}\}$ is bounded. Conversely if $\{c_{nn}\}$ is bounded, since $k \le s, C : \ell^k \to \ell^s$.

Combining these facts, $A : \ell^k \to \ell^s$ iff (3) and (4) are satisfied.

This completes the proof of the theorem. $\qquad\square$

COROLLARY 1.

Let $\{q_n\}$ be a positive sequence, $1 < k \le s < \infty$. Then $|C,1|_k \Rightarrow |\overline{N}, q_n|_s$ iff

(i) $$\left(\frac{n^{1+1/k-1/s}q_n}{Q_n} \right) = O(1)$$

and

(ii) $$\left(\sum_{n=m+1}^{\infty} \left(\frac{n^{1-1/s}q_n}{Q_n Q_{n-1}} \right)^s \right)^{1/s} \left(\sum_{v=1}^{m} |Q_v - (v+1)q_v|^{k^*} \left(\frac{1}{v} \right) \right)^{1/k^*} = O(1).$$

Proof. Set $p_n \equiv 1$ in Theorem 1. $\qquad\square$

COROLLARY 2.

Let $\{p_n\}$ be a positive sequence, $1 < k \le s < \infty$. Then $|\overline{N}, p_n|_k \Rightarrow |C,1|_s$ iff

(i) $$\frac{n^{(1/k)-(1/s)-1}P_n}{p_n} = O(1)$$

and

(ii) $$\left(\sum_{v=1}^{m} \left| v+1 - \frac{P_v}{p_v} \right|^{k^*} \left(\frac{1}{v} \right) \right)^{1/k^*} = O(m).$$

Proof. In Theorem 1, set $q_n \equiv 1$, to obtain condition (i). Then

$$\left(\frac{1}{n^{1/s}(n+1)} \right)^s = \frac{1}{n(n+1)^s},$$

$$I_1^s := \sum_{n=m+1}^{\infty} \frac{1}{n(n+1)^s} \ge \sum_{n=m+1}^{\infty} (n+1)^{-s-1}$$

$$> \int_{m+1}^{\infty} x^{-s-1} \, dx = (1/s)(m+1)^{-s}.$$

Therefore condition (ii) of Theorem 1 takes the form of condition (ii) of Corollary 2.

$\qquad\square$

COROLLARY 3.

Let $\{p_n\}, \{q_n\}$ be positive sequences. Then $|\overline{N}, p_n|_k \Rightarrow |\overline{N}, q_n|_k, k > 1$ iff

(i)
$$\frac{q_n P_n}{p_n Q_n} = O(1)$$

and

(ii)
$$\left(\sum_{n=m}^{\infty} \left(n^{1-1/k} \frac{q_n}{Q_n Q_{n-1}} \right)^k \right)^{1/k} \left(\sum_{v=1}^{m} \left| Q_v - \frac{q_v P_v}{p_v} \right|^{k^*} \left(\frac{1}{v} \right) \right)^{1/k^*} = O(1). \quad (12)$$

Proof. Corollary 3 comes from Theorem 1 by setting $s = k$. Formula (12) contains the complicated conditions referred to on page 3 of [2]. □

COROLLARY 4.

Let $k > 1$. Then $|C, 1|_k \Rightarrow |\overline{N}, p_n|_k$ iff

(i)
$$\frac{n p_n}{P_n} = O(1), \quad (13)$$

(ii)
$$\frac{P_n}{n p_n} = O(1) \quad (14)$$

hold.

Proof. Set $s = k$. Clearly the equivalence implies (13) and (14).

To prove the converse we must show that (13) and (14) imply conditions (ii) of Corollaries 1 and 2.

Using (13), with $s = k$,

$$\left(\frac{n^{1-1/k} p_n}{P_n P_{n-1}} \right)^k = \frac{n^{k-1} p_n^k}{(P_n P_{n-1})^k} = \frac{O(1) p_n}{P_n P_{n-1}^k}.$$

Therefore

$$\sum_{n=m+1}^{\infty} \left(\frac{n^{1-1/k} p_n}{P_n P_{n-1}} \right)^k = O(1) \sum_{n=m+1}^{\infty} \frac{p_n}{P_n P_{n-1}^k}$$

$$= \frac{O(1)}{P_m^{k-1}} \sum_{n=m+1}^{\infty} \frac{p_n}{P_n P_{n-1}} = \frac{O(1)}{P_m^k}.$$

From (14),

$$\sum_{v=1}^{m} \frac{|P_v - (v+1) p_v|^{k^*}}{v} = \sum_{v=1}^{m} \frac{P_v^{k^*}}{v} \left| 1 - \frac{(v+1) p_v}{P_v} \right|^{k^*}$$

$$= O(1) \sum_{v=1}^{m} \frac{P_v^{k^*}}{v} = O(1) \sum_{v=1}^{m} \frac{P_v}{v p_v} p_v P_v^{k^*-1}$$

$$= O(1) \sum_{v=1}^{m} p_v P_v^{k^*-1} \leq O(1) P_m^{k^*-1} \sum_{v=1}^{m} p_v$$

$$= O(1) P_m^{k^*}.$$

Therefore condition (ii) of Corollary 1 is satisfied.

From (14), using (13),

$$\sum_{v=1}^{m} \left| v+1 - \frac{P_v}{p_v} \right|^{k^*} \left(\frac{1}{v} \right) = \sum_{v=1}^{m} \frac{(v+1)^{k^*}}{v} \left| 1 - \frac{P_v}{(v+1)p_v} \right|^{k^*}$$

$$= O(1) \sum_{v=1}^{m} v^{k^*-1} = O(1) m^{k^*}.$$

Therefore condition (ii) of Corollary 2 is satisfied.   □

Corollary 4 is Theorem 5.1 of [5] since (1) and (2) are equivalent when conditions (13) and (14) hold.

We conclude by providing examples of weighted mean matrices satisfying Corollaries 1 and 2.

Example 1. For Corollary 1, choose $q_n = e^{-n}$. Then $Q_n = (1 - e^{-(n+1)})/(1-e)$, and

$$\frac{n^{1+1/k-1/s} q_n}{Q_n} = \frac{n^{1+1/k-1/s} e^{-n}(1-e)}{1 - e^{-(n+1)}}$$

$$= \frac{n^{1+1/k-1/s}(1-e)}{e^n - e^{-1}} \to 0 \quad \text{as} \quad n \to \infty,$$

and condition (i) is satisfied.

$$I_2^s := \sum_{n=m+1}^{\infty} \left(\frac{n^{1-1/s} q_n}{Q_n Q_{n-1}} \right)^s = \sum_{n=m+1}^{\infty} \left(\frac{n^{1-1/s} e^{-n}(1-e)^2}{(1 - e^{-(n+1)}(1-e^{-n}))} \right)^s$$

$$= (1-e)^{2s} \sum_{n=m+1}^{\infty} \frac{n^{s-1}}{(1-e^{-(n+1)})^s (e^n - 1)^s}$$

$$\leq \frac{(1-e)^{2s}}{(1-e^{-(m+2)})^s} \sum_{n=m+1}^{\infty} n^{s-1} e^{-ns} \left(\frac{e^{m+1}}{e^{m+1}-1} \right)$$

$$= O(1) \int_{m}^{\infty} x^{s-1} e^{-sx} \, dx$$

$$< O(1) \int_{m}^{\infty} x^{[s]} e^{-x} \, dx$$

$$= O(|P(m,[s])| e^{-m}),$$

where $P(m,[s])$ is a polynomial in m of degree $[s]$. Therefore

$$I_2 = O(|P(m,[s])|^{1/s} e^{-m/s}), \tag{15}$$

$$I_3^{k^*} := \sum_{v=1}^{m} |Q_v - (v+1)q_v|^{k^*} \frac{1}{v}$$

$$= \sum_{v=1}^{m} \frac{1}{v} \left| \frac{1 - e^{-(v+1)}}{1 - e} - (v+1)e^{-v} \right|^{k^*}$$

$$= \frac{1}{(1-e)^{k^*}} \sum_{v=1}^{m} \frac{1}{v} \left| 1 - e^{-(v+1)} - (v+1)e^{-v} + (v+1)e^{-(v+1)} \right|^{k^*}$$

$$= O(1) \sum_{v=1}^{m} \frac{1}{v} = O(\log m),$$

and

$$I_3 = O\left((\log m)^{1/k^*} \right). \tag{16}$$

Combining (16) and (17) gives condition (ii) of Corollary 1.

Example 2. For Corollary 2, use $p_n = 2^n$. Then $P_n = 2^{n+1} - 1$.

$$\frac{n^{1/k - 1/s - 1} P_n}{p_n} = \frac{n^{1/k - 1/s - 1}(s^{n+1} - 1)}{2^n}$$

$$= n^{1/k - 1/s - 1}(2 - 2^{-n}) \to 0 \quad \text{as} \quad n \to \infty$$

and condition (i) is satisfied.

$$I_4^{k^*} := \sum_{v=1}^{m} \left| v + 1 - \frac{P_v}{p_v} \right|^{k^*} \left(\frac{1}{v} \right) = \sum_{v=1}^{m} \left| v + 1 - \frac{(2^{v+1} - 1)}{2^v} \right|^{k^*} \frac{1}{v}$$

$$= \sum_{v=1}^{m} \left| v + 1 - 2 + 2^{-v} \right|^{k^*} \frac{1}{v} = \sum_{v=1}^{m} \frac{1}{v} |v - 1 + 2^{-v}|^{k^*}.$$

For $v \geq 1, 0 < v - 1 + 2^{-v} < v$. Therefore

$$I_4^{k^*} < \sum_{v=1}^{m} v^{k^* - 1} = O(m^{k^*}),$$

and condition (ii) of Corollary 2 is satisfied.

Acknowledgement

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